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"Matched" Polynomial
Least-Squares Fitting
and Application to
Real-Time Ballistic
Coefficient Estimation

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MASSACHUSETTS INSTITUTE OF TECHNOLOGY

LINCOLN LABORATORY

"MATCHED" POLYNOMIAL LEAST-SQUARES FITTING
AND APPLICATION TO REAL-TIME
BALLISTIC COEFFICIENT ESTIMATION

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ABSTRACT

A form of least-squares "spline function" fitting is
used to derive some general relationships that form the
basis of a new real-time ballistic coefficient estimation
technique.

"Matched" Polynomial Least-Squares Fitting

and

Application to Real-Time Ballistic Coefficient Estimation

Summary

A form of least-squares "spline-function" fitting[3] is presented where, in essence, a distinctly non-polynomial function defined over some gross inter- val is approximated by single polynomial expressions of degree M over each of contiguous sub-intervals with continuity conditions on the function up through the (M-1)'st derivative being imposed at the junction points (Fig. 1). Weighted least-squares fitting to given observational (radar) data then leads to linear algebraic equations for the parameter estimates instead of non-linear equations requiring solution by iteration. This technique leads to Eqs. (4), (8), and (9) of the analysis section. These relationships can be used for general weighted least-squares fitting of observational data where the "ordinary" polynomial approximation is not sufficiently accurate. Application of these relationships to real-time β estimation* appears promising and a procedure utilizing them is outlined in this paper.

Introduction

Standard least-squares polynomial fitting methods are easy to implement because the parameter estimates then satisfy linear algebraic equations which can be readily solved by standard matrix inversion techniques. However, long segments of re-entry trajectory are not generally well approximated by a single quadratic (or higher degree) polynomial expression. This is basically the reason why numerical integration of the differential equations of the trajectory followed by an iteration procedure and the ensuing mathematical complications inherent in a "Maximum-likelihood" approach, is frequently required in order to obtain high-

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* Ref. 1 discusses an alternate real-time method.

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precision answers. With a view toward real-time applications, we there-fore attempt to circumvent these complications by finding a more accurate functional representation than a single polynomial over the gross interval.

This is accomplished by subdividing the entire interval shown in Fig. 1 into equal sections, assuming a different polynomial expression of degree M over each section, and requiring the continuity of the function up through the $(M-1)$'st derivative at the intersection points T_2, T_3, ..., T_J, ..., T_p . This "matched" polynomial approach will yield accuracies some-where intermediate between ordinary least-squares fitting and the optimum iterative maximum-likelihood technique, and with considerably less compu-tation time. We now derive the necessary relationships and outline a real-time technique[*] that utilizes them.

Analysis:

In any region -- the J'th say -- of Fig. 1, a continuous differentiable function ''S(t)'' can be approximated by an Mth degree polynomial viz

$$S(t_i) = \sum_{m=0}^{M} \frac{S_J^{(m)}}{m!} (t_i - T_J)^m \qquad \text{for } T_J \leq t_i \leq T_{J+1} \qquad (1)$$

$$J = 1, 2, \ldots, p$$

where S_i is the value of the function $S(t)$ at $t = t_i$ in that region and $S_J^{(0)}$, $S_J^{(1)}$, $S_J^{(2)}$ ... are the values of $S(t)$, $\dot{S}(t)$, and $\ddot{S}(t)$ ... respectively at $t = T_J$.

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[*] The ''optimum'' real-time technique we define to be ''that technique which provides parameter estimates of 'SUFFICIENT' accuracy for the applications intended with a minimum of computation time and storage''.

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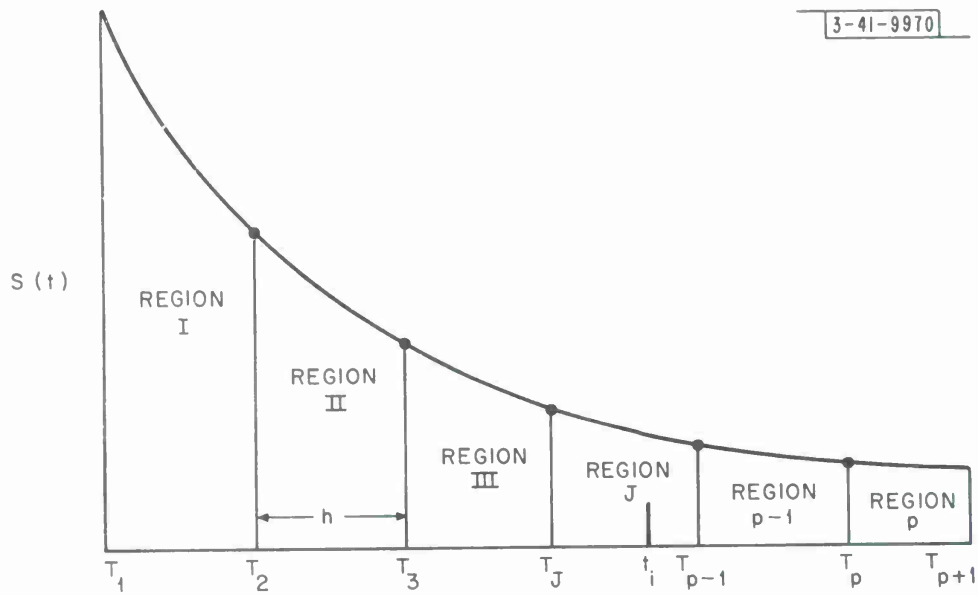
Fig. 1. Subregions for which different M'th-degree polynomial expressions apply.

A different M'th-degree polynomial of the type (1) applies over any region other than J . The data times t_i need not be equally spaced in what follows; however, it will be assumed that each region has an equal time span so that $h \equiv T_J - T_{J-1}$ is independent of J .

Continuity of S(t) up through the (M-1)'st derivative at the intersection point T_J of region J and (J-1) implies that

$$S_J^{(\mu)} = \sum_{m=\mu}^{M} \frac{S_{J-1}^{(m)} h^{m-\mu}}{(m-\mu)!} \qquad \mu = 0, 1, \ldots, (M-1) \qquad (2)$$

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The recursion relationship (2) can be shown to give

$$S_J^{(M-\nu)} = S_1^{(M-\nu)} + \sum_{k=1}^{J-1} \sum_{m=1}^{\nu} \frac{h^m}{m!} S_k^{(M+m-\nu)} \qquad \nu = 1, 2, \ldots, M \qquad (3a)$$
$$J = 2, 3, \ldots, p$$

Without the constraining relationships (3), there are $(M+1)$ p independent parameters to be determined in Eq. (1) by the least-squares fitting of noisy $S(t)$ data. However, the constraints (3) show that only $S_1^{(0)}$, $S_1^{(1)}$, ..., $S_1^{(M)}$, $S_2^{(M)}$, ..., $S_p^{(M)}$ are independent parameters $(M + p$ of them) to be determined by the fitting. $S_1^{(0)}$, $S_1^{(1)}$, ..., $S_1^{(M-1)}$ are the initial conditions at $t = T_1$ (corresponding to the initiation of a re-entry track) and the $S_K^{(M)}$ $K = 1, 2, \ldots, p$ are the M'th derivatives of the track in each region.

It can be shown by letting ν successively equal 1, 2, ..., that Eq. (3a) finally yields

$$S_J^{(M-\nu)} = \sum_{k=0}^{\nu-1} \frac{[(J-1)h]^k}{k!} S_1^{(M-\nu+k)} + \frac{h^\nu}{\nu!} \sum_{k=1}^{J-1} [(J-k)^\nu - (J-k-1)^\nu] S_k^{(M)} \qquad (3b)$$
$$\nu = 1, 2, \ldots, M$$
$$J = 2, 3, \ldots, p$$

In obtaining Eq. (3b), the following relationship was utilized:

$$\sum_{k=1}^{J-1} \sum_{k_1=1}^{J-1} \cdots \sum_{k_n=1}^{J-1} [\delta_{k_n < k_{n-1}} \delta_{k_{n-1} < k_{n-2}} \cdots \delta_{k_1 < k}] S_{k_n}^{(M)}$$

$$= \sum_{k=1}^{J-1} \frac{1}{n!} (J-k-1)(J-k-2) \ldots (J-k-n) S_k^{(M)}$$

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where, for example,

$$\delta_{k_1 < k} \equiv \begin{cases} 1 & \text{for } k_1 < k \\ 0 & \text{otherwise} \end{cases} .$$

Equations (3b) give the explicit dependence of the <u>dependent</u> parameters $S_J^{(M-\nu)}$, $J = 2, 3, \ldots, p$; $\nu = 1, 2, \ldots, M$ upon the independent parameters. The μ'th derivative of Eq. (1) at $t = t_i$ can now be written

$$S^{(\mu)}(t_i) = \sum_{k=0}^{M-1-\mu} S_1^{(k+\mu)} \frac{\left[h(J-1+\theta_{iJ})\right]^k}{k!} + \frac{h^{M-\mu}}{(M-\mu)!} \sum_{k=1}^{J-1} S_k^{(M)} \left\{ (J-k+\theta_{iJ})^{M-\mu} - \right.$$

$$\left. (J-k+\theta_{iJ} - 1)^{M-\mu} \right\} + \frac{(h\theta_{iJ})^{M-\mu}}{(M-\mu)!} S_J^{(M)} \qquad (4)$$

$$\theta_{iJ} \equiv \left(\frac{t_i - T_J}{h}\right) \qquad \begin{array}{l} T_J \leq t_i \leq T_{J+1} \\ J = 1, 2, \ldots, p \\ \mu = 0, 1, \ldots, M \end{array}$$

where the following relationship was noted:

$$\sum_{k=\mu}^{M-1} \frac{x^k y^{M-k}}{k! (M-k)!} = \frac{1}{M!} \left\{ (x+y)^M - x^M \right\} - \frac{1}{\mu!} \left\{ (x+y)^\mu - x^\mu \right\}$$

Notice that the only restrictions placed on the arbitrary function $S(t)$ were continuity and differentiability. Equation (4) is the desired analytical expression for the μ'th derivative where $\mu = 0$ corresponds to $S(t)$ itself. As is seen from Eq. (4), the function is represented over each of p contiguous segments by a different polynomial of the same degree M and continuity conditions on $S(t)$ up through the $(M - 1)$'st derivative are satisfied at all of the junction points.

From Eq. (4) we observe that

$$S(t_i) = \sum_{k=0}^{M-1} S_1^{(k)} \frac{\left[h(J-1+\theta_{iJ})\right]^k}{k!} + \frac{h^M}{M!} \sum_{k=1}^{J-1} S_k^{(M)} \left\{ (J-k+\theta_{iJ})^M - \right.$$

$$\left. (J-k+\theta_{iJ} - 1)^M \right\} + \frac{(h\theta_{iJ})^M}{M!} S_J^{(M)} \qquad (5a)$$

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$$\frac{\partial S(t_i)}{\partial S_1^{(\mu)}} = \frac{\left[h(J-1+\theta_{iJ})\right]^{\mu}}{\mu!} \qquad \mu = 0,\ 1,\ 2,\ \ldots,\ (M-1) \quad (5b)$$

$$\frac{\partial S(t_i)}{\partial S_K^{(M)}} = \frac{h^M}{M!}\left\{\delta_{K,J}\,\theta_{iJ}^M + \delta_{K<J}\left[(J-K+\theta_{iJ})^M - (J-K-1+\theta_{iJ})^M\right]\right\} \qquad (5c)^{\dagger}$$

$$J,\ K = 1,\ 2,\ \ldots,\ p$$

$$T_J \le t_i \le T_{J+1}$$

Consider the quantity $\{S(t_i) - S_i^{*}\}$ where S_i^{*} is given noisy experimental data at $t = t_i$ with variance σ_i^2. Physically, this quantity represents the deviation of the data from the calculated function at time t_i in region J. We therefore wish to minimize

$$\Lambda = \sum_{J=1}^{p}\ \sum_{i=N_{J-1}+1}^{N_J} \left\{\frac{S(t_i) - S_i^{*}}{\sigma_i}\right\}^2 , \qquad (6)$$

where N_J is the total number of points up to the value T_{J+1}.

Differentiating Λ partially with respect to $S_1^{(0)}, S_1^{(1)}, \ldots, S_1^{(M-1)}$, $S_K^{(M)}$, $K = 1,\ 2,\ \ldots,\ p$ and equating the results to zero gives

$$\sum_{J=1}^{p}\ \sum_{i=N_{J-1}+1}^{N_J} \left\{\frac{S(t_i) - S_i^{*}}{\sigma_i^2}\right\}\ \frac{\left[h(J-1+\theta_{iJ})\right]^{\mu}}{\mu!} = 0 \qquad (7a)$$

$$\mu = 0,\ 1,\ \ldots,\ (M-1)$$

$$\sum_{J=1}^{p}\ \sum_{i=N_{J-1}+1}^{N_J} \left\{\frac{S(t_i) - S_i^{*}}{\sigma_i^2}\right\}\ \frac{h^M}{M!}\left\{\delta_{K,J}\,\theta_{iJ}^M + \right.$$

$$\left. + \delta_{K<J}\left[(J-K+\theta_{iJ})^M - (J-K+\theta_{iJ}-1)^M\right]\right\} = 0 \quad K = 1,\ \ldots,\ p \qquad (7b)$$

$\dagger$ Where $\delta_{K,J}$ is the Kronecker delta.

Equations (7) represent (M + p) simultaneous linear algebraic equations for the (M + p) independent parameter values. In matrix notation, Eqs. (7) become

$$AX = B \tag{8a}$$

where A turns out to be symmetric and

$$
A = \begin{bmatrix}
a_{11} & a_{12} \cdots\cdots & a_{1,\,p+M} \\
a_{21} & a_{22} \cdots\cdots & a_{2,\,p+M} \\
\vdots & & \vdots \\
& & \\
& & \\
a_{p+M,\,1} & a_{p+M,\,2} \cdots\cdots & a_{p+M,\,p+M}
\end{bmatrix}
\qquad
X = \begin{bmatrix}
S_1^{(0)} \\
S_1^{(1)} \\
\vdots \\
S_1^{(M-1)} \\
S_1^{(M)} \\
\vdots \\
S_p^{(M)}
\end{bmatrix}
\qquad
B = \begin{bmatrix}
b_1 \\
b_2 \\
\vdots \\
\\
\\
b_{p+M}
\end{bmatrix}
\tag{8b}
$$

There remains only the determination of $a_{\alpha,\,\beta}$ and b_{α} $\alpha,\ \beta = 1,\ 2,\ \ldots,$ (p+M) .

Equations (7) and (5a) give

$$
a_{\mu+1,\,K+1} = \frac{h^{\mu+K}}{\mu!\,K!} \sum_J \sum_i \frac{1}{\sigma_i^2} \left(J-1+\theta_{iJ}\right)^{\mu+K}
\qquad \mu, K = 0, 1 \ldots, M-1 \tag{9a}
$$

and $a_{K+1,\,\mu+1} = a_{\mu+1,\,K+1}.$

$$
a_{\mu+1,\,M+K} = \frac{h^{M+\mu}}{M!\,\mu!} \sum_J \sum_i \frac{\left(J-1+\theta_{iJ}\right)^{\mu}}{\sigma_i^2} \left\{ \delta_{K,\,J}\,\theta_{iJ}^M + \right.
$$

$$
\left. \delta_{K<J} \left[\left(J-K+\theta_{iJ}\right)^M - \left(J-K-1+\theta_{iJ}\right)^M \right] \right\} \tag{9b}
$$

$$\mu = 0, 1,\ \ldots,\ M-1$$

$$K = 1,\ 2,\ \ldots,\ p$$

and $a_{M+K,\,\mu+1} = a_{\mu+1,\,M+K}.$

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$$a_{M+K,\,M+L} = \frac{h^{2M}}{(M!)^2} \sum_J \sum_i \left\{ \delta_{K,J}\, \theta_{iJ}^M + \delta_{K<J} \left[(J-K+\theta_{iJ})^M - (J-K-1+\theta_{iJ})^M \right] \right\}$$

$$\times \left\{ \delta_{L,J}\, \theta_{iJ}^M + \delta_{L<J} \left[(J-L+\theta_{iJ})^M - (J-L-1+\theta_{iJ})^M \right] \right\} \frac{1}{\sigma_i^2} \quad (9c)$$

$$K, L = 1, 2, \ldots, p$$

and $a_{M+L,\,M+K} = a_{M+K,\,M+L}$.

Thus, the entire A matrix is symmetric!

The elements of the B matrix are given by

$$b_{\mu+1} = \frac{h^\mu}{\mu!} \sum_J \sum_i \frac{S_i^*}{\sigma_i^2} (J-1+\theta_{iJ})^\mu \qquad \mu = 0, 1, \ldots, M-1 \qquad (9d)$$

$$b_{M+K} = \frac{h^M}{M!} \sum_J \sum_i \frac{S_i^*}{\sigma_i^2} \left\{ \delta_{K,J}\, \theta_{iJ}^M + \delta_{K<J} \left[(J-K+\theta_{iJ})^M - (J-K-1+\theta_{iJ})^M \right] \right\}$$

$$(9e)$$

$$K = 1, 2, \ldots, p$$

where in (9a)-(9e) it is understood that the summation limits on J are 1 to p
and the limits on i are $N_{J-1}+1$ to N_J.

Equations (4), (8), and (9) are the relationships required for "matched"
polynomial least-squares fitting of non-polynomial data by means of multiple
M'th-degree polynomials in segments, and constrained to satisfy continuity
conditions through the (M-1) derivative.[†] The parameter estimates via
Eqs. (8) are linear and the matrix elements are simple summations of simple
algebraic expressions ⌊Eqs. (9)⌋. The special case of ordinary least-squares
polynomial fitting about $t = T_1$ results by setting p = 1 in Eqs. (5a) and (9).
It seems clear that computation time and storage should be nearly minimal
and whether the technique has sufficient accuracy compared to iterative
maximum-likelihood estimation becomes the chief concern. Adaptation to
real-time β estimation will now be discussed.

[†] Appendix I particularizes these formulas to the case of quadratic poly-
nomials (M=2). "Matched cubics" (M=3) are the smallest degree polynomials
for which the second (and lower order) derivatives are continuous throughout
the gross interval of interest. Consequently, use of M=3 in the above formulas
seems particularly promising for the trajectory applications that follow.

From Newtonian dynamics[2]

$$\ddot{S} = f - \frac{g}{\beta(t)} \qquad\qquad g = \frac{\rho v \dot{S}}{2} \qquad\qquad (10)$$

where $\ddot{S}$ is the second derivative of range, ρ the air weight density at time t, v the missile velocity, $\dot{S}$ the range rate, and β the ballistic coefficient. Physically, g/β is the component of the drag force along the radar line of sight. The f function contains components of acceleration along the line of sight due to angular rates, gravity, and earth rotation effects. No Doppler data will be assumed in what follows. (Inclusion of Doppler information will be discussed later.)

The f and g functions in Eq. (10) contain no second derivatives.[†] The smooth range, angle values, their rates, and $\ddot{S}$ all at time t are to be determined from the radar data by means of Eqs. (8) and (9). For $M = 2$, since $\ddot{S}$ has different constant values for each region, Eq. (10) solved for β, represents a "staircase" ballistic coefficient history. For $M > 2$, $\ddot{S}$ is continuous and consequently, β is also. The following outlines a real-time β method.

1. A few samples, say "i_1", of radar metric data are obtained at a high re-entry altitude. Times T_J, $J = 1, 2, \ldots$ have been defined a priori and for convenience it will be assumed that at least one radar sample will be obtained in each region (Fig. 1).

2. Equations (8) and (9) with $p = 1$, for example, are solved with S_i^* representing the range radar data with $\sigma_i = \sigma_i^{RANGE}$, and $i = 1, 2, \ldots, i_1$. Thus, we obtain the first estimate of S_1, $\dot{S}_1$, $\ddot{S}_1 \ldots$. Similarly, Eqs. (8) and (9), with S_i^* representing the elevation data and then the azimuth data, are solved to give the first estimates of E_1, $\dot{E}_1$, $\ddot{E}_1 \ldots$ and then A_1, $\dot{A}_1$, $\ddot{A}_1 \ldots$.

[†] In particular, the absence of angle second derivatives makes this equation particularly suitable for calculating the ballistic coefficient because radar angle errors are ordinarily much more serious than range and Doppler errors.

3.   Equations (4) give S_i, $\dot{S}_i$, $\ddot{S}_i$, E_i, $\dot{E}_i$, A_i, $\dot{A}_i$, and Eq. (10) can be solved for $\beta(t_i)$, yielding the first estimate of β in region 1 at $t = t_i$.

4.   As the next set of one (or more) radar data samples are received, giving a total of $i_2 = i_1 + \Delta i_1$, two distinct cases arise:

a) i_2 lies within the previous or $p = 1$ region above, or

b) i_2 lies in the $p \geq 2$ region.

For case (a) the changes in the matrix elements $\Delta a_{\alpha\beta}$ and Δb_α, α, $\beta = 1, 2, \ldots, (M + 1)$ are calculated from Eqs. (9) with $p = 1$ and the index i ranging from $i_1 + 1$, i_2. It is important to notice that only the new data samples are included in these summations and that the old summations need not be recalculated. The new matrix elements, "$\nu + 1$", are given in terms of the old "ν" as

$$a_{\alpha\beta}^{(\nu+1)} = a_{\alpha\beta}^{(\nu)} + \Delta a_{\alpha\beta}$$

$$b_\alpha^{(\nu+1)} = b_\alpha^{(\nu)} + \Delta b_\alpha \tag{11}$$

and the parameter estimates are updated via Eq. (8a)

$$X = A^{-1}B \quad . \tag{12}$$

Equation (4) is then used to calculate S_i, $\dot{S}_i$, $\ddot{S}_i$, E_i, $\dot{E}_i$, A_i, $\dot{A}_i$, and the new β value, $\beta(t_i)$, is calculated from Eq. (10).

For case (b), assume for convenience that i_2 lies in the second region $p = 2$. The changes in the matrix elements $\Delta a_{\alpha\beta}$ and Δb_α, α, $\beta = 1, 2, \ldots, (M + 2)$ are calculated using only the new radar data and Eqs. (9). The summation limits are $p = 1$ and $i = i_1 + 1$, N_1 for the first region; and $p = 2$ and $i = N_1 + 1$, i_2 for the second. The new values of the matrix elements are calculated from Eq. (11) and the updated parameter values from Eq. (12). Then, S_i, $\dot{S}_i$ $\ldots$, $\beta(t_i)$ are calculated as for case (a) above. We note, however, that this case (b) not only updates all of the previous parameter estimates, but furnishes the first estimates for β in region 2, requiring, however, the inversion of a $(M + 2) \times (M + 2)$ matrix instead of a $(M + 1) \times (M + 1)$ as in case (a).

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5.   Beyond steps (3) and (4), continuation of the real-time calculation should be evident. For practicality, a limit on the largest size matrix to be inverted must be imposed. Such a limit might in practice require some additional steps which need not concern us here.

It was noted that the A and B matrices involve simple summations over the available data points, and as more data points are obtained, these summations need only be updated rather than recalculated. The solution matrix X for the latest parameter estimates, however, do require recalculation. Attention is now focused on this calculation.

Assume that $X^{(\nu)}$ has been determined and includes data points recently acquired in region J . New data points are then acquired

(a)   entirely in the same region J (as in case (a) above, or

(b)   at least one of the new points falls in region (J + 1) (as in case (b) above) .

Letting $A^{(\nu+1)}$, $B^{(\nu+1)}$, and $X^{(\nu+1)}$ denote the new A, B, and X matrices respectively, we have

$$A^{(\nu+1)} = A^{(\nu)} + (\Delta A)$$
$$B^{(\nu+1)} = B^{(\nu)} + (\Delta B)$$
$$X^{(\nu+1)} = X^{(\nu)} + (\Delta X) \qquad . \tag{13}$$

The matrices with superscript (ν) in Eqs. (13) are known from previous calculation. The (ΔA) and (ΔB) matrices defined by Eqs. (11) are updated using Eqs. (9). Thus, only ΔX and consequently $X^{(\nu+1)}$ are to be redetermined.

From Eqs. (12) and (13) there follows

$$\Delta X = [A^{(\nu+1)}]^{-1}[\Delta B - (\Delta A) X^{(\nu)}] \qquad . \tag{14}$$

Equation (14) will apply for both cases (a) and (b) providing $X^{(\nu)}$, for example, is defined as a (J + 1) + M column vector with the last element identically zero for case (a) or identically the (J + 1 + M) independent variable if case (b); similarly with the other matrices in Eq. (14).

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The new parameter estimates are thus determined from Eq. (14) in terms of the old parameters $X^{(\nu)}$, the old A matrix $A^{(\nu)}$, and the changes in the A and B matrices. Computation time and storage are clearly a minimum except possibly for the inversion of the $A^{(\nu+1)}$ matrix that is required with every acquisition of new data.

When good Doppler data is available, the same formulas and methods presented are used with the required changes in notation. The $\dot{S}(t)$ and $\ddot{S}(t)$ histories previously calculated from the range data are merely replaced by the values calculated from the Doppler information. For $M = 2$, the β history calculated using Doppler information is clearly continuous.

The $A^{(\nu)}$ matrix of Eq. (14) grows by one row and one column whenever data is acquired in a new region, which restricts the number of regions that can be utilized in applications of this technique.

HS:cm

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ACKNOWLEDGEMENT

APPENDIX I

For convenience, we particularize the formulas derived in the text to matched quadratic polynomials and list the results here (M = 2).

Equation (4) of the text becomes

$$S(t_i) = S_1 + h(J - 1 + \theta_{iJ})\dot{S}_1 + \frac{h^2}{2}\theta_{iJ}^2 \ddot{S}_J + h^2 \sum_{k=1}^{J-1}(J - k - \frac{1}{2} + \theta_{iJ})\ddot{S}_k \qquad (A-1)$$

$$\dot{S}(t_i) = \dot{S}_1 + h(\theta_{iJ}\ddot{S}_J + \sum_{k=1}^{J-1}\ddot{S}_k)$$

$$\theta_{iJ} = \frac{t_i - T_J}{h} \qquad\qquad T_J \le t_i \le T_{J+1}$$

$$J = 1, 2, \ldots, p$$

The matrix elements in conjunction with Eqs. (8a) and (8b) reduce to

$$a_{11} = \sum_{J=1}^{p} \sum_{i=N_{J-1}+1}^{N_J} 1/\sigma_i^2$$

$$a_{21} = \sum_J \sum_i h(J - 1 + \theta_{iJ})/\sigma_i^2$$

$$a_{22} = h^2 \sum_J \sum_i (J - 1 + \theta_{iJ})^2/\sigma_i^2$$

$$a_{2+K, 1} = h^2 \sum_J \sum_i [\frac{\delta_{J,K}}{2}\theta_{iK}^2 + \delta_{J>K}(J - K - \frac{1}{2} + \theta_{iJ})]/\sigma_i^2$$

$$K = 1, 2, \ldots, p$$

$$a_{2+K, 2} = h^3 \sum_J \sum_i (J - 1 + \theta_{iJ})[\frac{\delta_{J,K}}{2}\theta_{iK}^2 + \delta_{J>K}(J - K - \frac{1}{2} + \theta_{iJ})]/\sigma_i^2$$

$$a_{2+K,\,2+L} = h^4 \sum_J \sum_i [\frac{\delta_{J,K}}{2}\theta_{iK}^2 + \delta_{J>K}(J-K-\frac{1}{2}+\theta_{iJ})][\frac{\delta_{J,L}}{2}\theta_{iL}^2 + \delta_{J>L}(J-L-\frac{1}{2}+\theta_{iJ})]/\sigma_i^2$$

$$K = 1,\,2,\,\ldots,\,p \quad L = 1,\,2,\,\ldots\,p$$

$$a_{KL} = a_{LK} \qquad\qquad L,\,K = 1,\,2,\,\ldots,\,2+p$$

$$b_1 = \sum_{J=1}^{p}\ \sum_{i=N_{J-1}+1}^{N_J} S_i^*/\sigma_i^2$$

$$b_2 = h \sum_J \sum_i (J - 1 + \theta_{iJ})\,S_i^*/\sigma_i^2$$

$$b_{2+K} = h^2 \sum_J \sum_i [\frac{\delta_{J,K}}{2}\theta_{iK}^2 + \delta_{J>K}(J-K-\frac{1}{2}+\theta_{iJ})]S_i^*/\sigma_i^2 \qquad\qquad (A-2)$$

where
$$K = 1,\,2,\,\ldots,\,p$$

$$\delta_{J>K} = \begin{cases} 1 & \text{for } J > K \\ 0 & \text{otherwise} \end{cases} \qquad\qquad \delta_{J,K} = \begin{cases} 1 & \text{for } J=K \\ 0 & \text{otherwise} \end{cases}$$

An alternative to standard formulas for performing numerical integration is obtained by integrating Eq. (A-1). The result is

$$\int_{T_1}^{T_{p+1}} S(t)\,dt = hp\,\{S_1 + \dot{S}_1 \frac{(hp)}{2!} + \frac{h^2}{3!\,p}\sum_{k=1}^{p}[3(p-k+1)(p-k)+1]\ddot{S}_k\} \qquad\qquad (A-3)$$

$$p = 1,\,2,\,3,\,\ldots$$

Referring to Fig. 1, S_1 and $\dot{S}_1$ are the values of the integrand and its first derivative at $t = T_1$ and $\ddot{S}_k$ is the value of the second derivative at T_k $k = 1,\,2,\,\ldots,\,p$. The formula becomes exact as $p \to \infty$, $h \to 0$.

A-2

Integration of Eq. (5a) followed by summation over J gives the more general quadrature formula

$$
\int_{T_1}^{T_{p+1}} S(t)\, dt = \sum_{k=0}^{M-1} S_1^{(k)} \frac{(hp)^{k+1}}{(k+1)!} + \frac{h^{M+1}}{(M+1)!} \sum_{k=1}^{p} S_k^{(M)} \times
$$

$$
\left\{ (p-k+1)^{M+1} - (p-k)^{M+1} \right\} \quad p = 1,\, 2,\, \ldots \tag{A-4}
$$

The integrand $S(t)$ can be evaluated at various points t_i. The matrix elements of Eqs. (9a-9e) can be calculated, and the independent parameters appearing in the right-hand side of Eq. (A-4) are obtained via Eq. (12).

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13. ABSTRACT

A form of least-squares "spline function" fitting is used to derive some general relationships that form the basis of a new real-time ballistic coefficient estimation technique.

14. KEY WORDS

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